

W39. Let $a, b, c \in \mathbb{R}$. Solve the equations:

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0 \quad (1)$$

$$x^3 - 3ax^2 + (3a^2 - b^2 - c^2 - bc)x - a^3 + ab^2 + ac^2 - b^2c - bc^2 + abc = 0 \quad (2)$$

Daniel Sitaru

Solution by Arkady Alt, San Jose, California, USA.

1. $x^3 - 3x^2a + 3xa^2 - a^3 - 3b^2x + 3b^2a - 2b^3 = (x - a)^3 - 3b^2(x - a) - 2b^3 =$
 $t^3 - 3b^2t - 2b^3$, where $t := x - a$ and $t^3 - 3b^2t - 2b^3 = t^3 - b^2t - 2b^2t - 2b^3 =:$
 $(t + b)(t^2 - bt) - 2b^2(t + b) = (t + b)(t^2 - bt - 2b^2) = (t + b) \cdot (t + b)(t - 2b) =$
 $(t + b)^2(t - 2b)$ then $(x - a)^3 - 3b^2(x - a) - 2b^3 = (x - a - 2b)(x - a + b)^2$
 and, therefore, $x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0 \Leftrightarrow$

$$(x - a - 2b)(x - a + b)^2 = 0 \Leftrightarrow \begin{cases} x = a + 2b \\ x = a - b \end{cases}.$$

2. $x^3 - 3ax^2 + (3a^2 - b^2 - c^2 - bc)x - a^3 + ab^2 + ac^2 - b^2c - bc^2 + abc =$
 $(x - a)^3 - b^2(x - a) - c^2(x - a) - bc(x - a) - bc(b + c) =$
 $t^3 - t(b^2 + bc + c^2) - bc(b + c)$, where $t := x - a$ and
 $t^3 - t(b^2 + bc + c^2) - bc(b + c) = t^3 - t(b + c)^2 + bc(t - (b + c)) =$
 $(t - b - c)(t^2 - (b + c)t + bc) = (t - b - c)(t - b)(t - c)$ then
 $x^3 - 3ax^2 + (3a^2 - b^2 - c^2 - bc)x - a^3 + ab^2 + ac^2 - b^2c - bc^2 + abc = 0 \Leftrightarrow$

$$(x - a - b - c)(x - a - b)(x - a - c) = 0 \Leftrightarrow \begin{cases} x = a + b + c \\ x = a + b \\ x = a + c \end{cases}$$